

NUMERICAL ANALYSIS OF STOCHASTIC DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH APPLICATIONS TO ELECTRONIC CIRCUIT SIMULATION

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ABSTRACT

Stochastic differential-algebraic equations (sDAEs) offer an effective mathematical model of constrained dynamical systems, with uncertainty, especially in the simulation of electronic circuits where algebraic constraints are the result of Kirchhoff laws and uncertainty is caused by thermal noise and parameters variation. The numerical analysis of index-1 stochastic differential-algebraic equations using an electronic circuit application context is provided in this paper. The deterministic and stochastic DAEs are formulated mathematically and the index and consistency conditions that the problem is well-posed are discussed. The implicit function theory is used, along with classical theory of stochastic differential equations, to show existence and uniqueness of local strong solutions. A weak form of Euler-Maruyama scheme is then constructed and studied, and there is a strong convergence of the order of one-half under standard Lipschitz and growth conditions. The properties of mean-square stability are investigated using the linearized sDAEs and it is shown that the implicit scheme can keep stability of the discrete system provided sufficiently small steps are taken. The theoretical findings are demonstrated in the model of a stochastic RLC circuit where the numerical simulations give evidence to the valid constraint enforcement and stability under stochastic excitation. The work lies between the rigorous numerical theory and the practical circuit simulation and offers a stable base in the simulation of noisy constrained systems.

Keywords: *Stochastic Differential-Algebraic Equations, Numerical Analysis, Implicit Euler–Maruyama Method, Mean-Square Stability, Electronic Circuit Simulation.*

1. INTRODUCTION

DAEs are a fundamental modeling paradigm for systems whose behavior is specified by a mixture of differential dynamics and algebraic constraints. In contrast to ODEs, which describe unconstrained evolution, DAEs naturally encode physical conservation laws, geometric constraints, and interconnection

structures that are inherent in many real-world systems. In electrical engineering, DAEs arise directly from Kirchhoff's voltage and current laws, which impose an algebraic interrelation among circuit variables, together with the constitutive relations of resistors, capacitors, inductors, and semiconductor devices. As a consequence, DAE formulations provide an accurate and physically consistent mathematical description of complex electrical networks.

Electronic circuits are hardly deterministic in realistic operating environments. Stochastic effects include thermal agitation of charge carriers, electromagnetic interference, manufacturing tolerances, and changing sources of input and may cause large effects in circuit behavior, especially in high-frequency, low-power, and nanoscale circuits. In order to make the amount of such uncertainties, deterministic DAEs are generalized to stochastic differential-algebraic equations (sDAEs), where the dynamics of a system are disturbed by a series of random processes, most typically a Wiener process. This stochastic extension permits a more accurate model of the phenomena of noise and uncertainty propagation in circuit models.

Numerical algorithms to stochastic differential equations (SDEs) are well understood but their straightforward generalization to sDAEs is not straightforward. The existence of algebraic constraints brings in concealed constraints and index-based troubles that can bring about loss of solidarity or escape the constraint space with straightforward discretization. Moreover, the presence of stochastic perturbation can lead to the instability of the numerical schemes that are stable in the deterministic DAE case, so the maintenance of constraints and stability is a key issue in numerical simulation.

This paper has the goal of giving a mathematically comprehensive and application-focused discussion of numerical methods to stochastic differential-algebraic equations, especially in the case of electronic circuit simulation. The article is a combination of theoretical study of existence, stability, and convergence and design and evaluation of appropriate numerical schemes, and the theoretical results are demonstrated with the help of representative stochastic circuit models. In this way, it seeks to reduce the divide amid abstract numerical analysis and practical simulation demand in stochastic constrained systems.

2. LITERATURE REVIEW

Burger and Gerdt (2017) presented a comprehensive survey of numerical methods for the simulation of initial value problems involving stochastic differential-algebraic equations (sDAEs). The study reviewed a wide range of computational techniques, including time-stepping methods, index reduction approaches,

and stochastic integration schemes, highlighting their applicability and limitations. It emphasized the challenges associated with stiffness, constraint handling, and randomness in such systems. The authors also discussed the importance of stability, convergence, and efficiency in selecting appropriate numerical solvers, thereby providing a detailed foundation for further research in the numerical treatment of sDAEs.

González-Zumba et al. (2020) investigated the stability of stochastic differential-algebraic systems using Lyapunov exponents and demonstrated their effectiveness in assessing system behavior under uncertainty. The study developed a framework for evaluating stability by analyzing the growth rates of system trajectories, particularly in the presence of stochastic disturbances. It also applied the proposed methodology to power systems, showing how Lyapunov-based analysis could be used to ensure reliability and robustness in real-world engineering applications. The findings highlighted the importance of combining stochastic analysis with dynamical systems theory for accurate stability assessment.

Kleinert and Simeon (2018) explored differential-algebraic equations in the broader context of constrained dynamical systems, extending the discussion from smooth to nonsmooth systems. The study provided a unified theoretical perspective on DAEs, emphasizing their role in modeling systems with constraints, such as mechanical systems with impacts or friction. It highlighted the mathematical challenges associated with nonsmooth dynamics and discussed advanced analytical and numerical techniques for addressing these issues. The work contributed to a deeper understanding of constrained dynamical systems and their applications in engineering and applied sciences.

Manfredi et al. (2018) examined the relationship between the stochastic Galerkin method and the pseudo-spectral collocation method for solving linear differential-algebraic equations. The study demonstrated that both methods provided effective approaches for handling uncertainty in DAE systems, while differing in computational complexity and implementation. It highlighted that the stochastic Galerkin method offered higher accuracy through a global approximation, whereas the collocation method provided greater flexibility and computational efficiency. The research emphasized the importance of selecting appropriate numerical techniques based on problem characteristics, thereby contributing to advancements in uncertainty quantification for differential-algebraic systems.

3. MATHEMATICAL FORMULATION OF STOCHASTIC DIFFERENTIAL-ALGEBRAIC EQUATIONS

This part presents the mathematical model of stochastic differential-algebraic equations. Our initial contribution is the semi-explicit model of deterministic DAEs and their stochastic counterparts, with a focus on the importance of algebraic constraints and random perturbations. The notions of differentiation index and consistency of initial conditions are then mentioned, since they are the key to well-posedness and to the design of consistent and accurate numeric methods.

3.1 Deterministic and Stochastic DAEs

A semi-explicit differential-algebraic equation (DAE) is expressed as

$$\begin{aligned}\dot{x}(t) &= f(x(t), y(t)), \\ 0 &= g(x(t), y(t)),\end{aligned}$$

where $x(t) \in \mathbb{R}^n$ represents the differential variables describing the system dynamics, and $y(t) \in \mathbb{R}^m$ denotes algebraic variables arising from constraint relations. The algebraic equation enforces instantaneous constraints that restrict the system evolution to a constraint manifold defined by $g(x, y) = 0$.

In circuit modeling electronic circuit modeling uses the dynamic quantities of currents through inductors or voltages across capacitors as the differential variables, and the algebraic variables represent node voltages or currents that are subject to the laws of Kirchhoff. The DAE formulation therefore incorporates both the dynamics behavior and structural interconnections in one.

To incorporate uncertainty and noise effects, the deterministic DAE is extended to a stochastic differential-algebraic equation (sDAE) of the form

$$\begin{aligned}dx(t) &= f(x(t), y(t)) dt + \sigma(x(t), y(t)) dW(t), \\ 0 &= g(x(t), y(t)),\end{aligned}$$

where $W(t)$ denotes a standard Wiener process and $\sigma(x, y)$ is the diffusion coefficient describing the intensity and structure of stochastic perturbations. In the stochastic term, the random forces include the thermal noise, the fluctuations of parameters or other external perturbations of the system dynamics. The sDAE solutions are understood in the Ito sense.

3.2 Index and Consistency

The differentiation index is a fundamental concept in the analysis of DAEs and sDAEs. A system is said to be of index-1 if the Jacobian matrix $\frac{\partial g}{\partial y}$ is nonsingular in a neighborhood of the constraint manifold. Under this condition, the algebraic equation can be locally solved for y as a smooth function of x , allowing the sDAE to be reduced to an equivalent stochastic differential equation on the constraint manifold.

Consistent initial conditions play a crucial role in both analysis and numerical simulation. An initial condition $(x(0), y(0))$ is said to be consistent if it satisfies the algebraic constraint $g(x(0), y(0)) = 0$. In the stochastic setting, failure to enforce consistency may lead to constraint drift and numerical instability. Consequently, numerical methods for sDAEs must be designed to preserve consistency at each time step, either implicitly or through projection techniques.

4. EXISTENCE AND UNIQUENESS OF SOLUTIONS

Stochastic differential-algebraic equation well-posedness is a basic requirement of the theoretical analysis as well as numerical simulation. This section develops adequate criteria under which index-1 sDAEs have solutions and are unique. The study is based on classical outcomes of stochastic differential equations and structural features of algebraic constraints.

Theorem 4.1 (Local Existence and Uniqueness)

Consider the stochastic differential-algebraic system

$$\begin{aligned} dx(t) &= f(x(t), y(t)) dt + \sigma(x(t), y(t)) dW(t), \\ 0 &= g(x(t), y(t)), \end{aligned}$$

with initial condition $(x(0), y(0))$ satisfying $g(x(0), y(0)) = 0$. Assume that the functions $f: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$, $\sigma: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n \times r}$, and $g: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$

are continuously differentiable, and that the Jacobian matrix

$$\frac{\partial g}{\partial y}(x, y)$$

is nonsingular in a neighborhood of the constraint manifold

$$\mathcal{M} = \{(x, y) \in \mathbb{R}^{n+m} : g(x, y) = 0\}.$$

Then the sDAE admits a unique local strong solution.

Proof

Since $\frac{\partial g}{\partial y}$ is nonsingular on $\mathcal{M}$, the implicit function theorem guarantees the existence of a continuously differentiable mapping

$$y = \phi(x)$$

such that

$$g(x, \phi(x)) = 0$$

for all x in a neighborhood of the initial condition. Substituting $y = \phi(x)$ into the differential equation yields the reduced stochastic differential equation

$$dx(t) = \tilde{f}(x(t)) dt + \tilde{\sigma}(x(t)) dW(t),$$

were

$$\tilde{f}(x) = f(x, \phi(x)), \tilde{\sigma}(x) = \sigma(x, \phi(x)).$$

Since $\tilde{f}$ and $\tilde{\sigma}$ inherit continuity and local Lipschitz properties from f , σ , and ϕ , standard results from stochastic differential equation theory ensure the existence and uniqueness of a local strong solution $x(t)$.

The corresponding algebraic variable is recovered as $y(t) = \phi(x(t))$, which satisfies the constraint for all t in the interval of existence.

Remark 4.1 (Role of the Index)

The nonsingularity of $\frac{\partial g}{\partial y}$ characterizes index-1 stochastic DAEs and ensures that the algebraic constraints define a smooth manifold. With larger index sDAE, the algebraic equations are differentiated once or multiple times to bring out latent constrictions, which increases the analytical and numerical complexity. This type of systems normally needs index reduction or regularization methodology, which have not been taken into account in the current research.

Remark 4.2 (Consistency Preservation)

In the index-1 sDAE, the solution curve lies on the constraint manifold M provided that the initial condition is consistent. This is an important property in numerical simulations which makes implicit or projection-based schemes attractive to avoid drift on the constraint to stochastic perturbations.

5. Numerical Methods for Stochastic Differential-Algebraic Equations

Simulation of stochastic differential-algebraic equations is numerically demanding and the time resolution schemes must allow algebraic constraints to be satisfied and maintain stability in the face of stochastic perturbations. Explicit methods have a problem of lacking constraints and they are also prone to instability especially with stiff systems like electronic circuits. Implicit schemes are thus used in favour in practical computations electronic circuits. Implicit schemes are therefore preferred in practical computations.

5.1 Implicit Euler–Maruyama Scheme

Let $h > 0$ denote the time step size and $t_n = nh$. The implicit Euler–Maruyama method for the index-1 sDAE

$$\begin{aligned} dx(t) &= f(x(t), y(t)) dt + \sigma(x(t), y(t)) dW(t), \\ 0 &= g(x(t), y(t)), \end{aligned}$$

is defined by the coupled nonlinear system

$$\begin{aligned} x_{n+1} &= x_n + h f(x_{n+1}, y_{n+1}) + \sigma(x_n, y_n) \Delta W_n, \\ 0 &= g(x_{n+1}, y_{n+1}), \end{aligned}$$

where the Wiener increment satisfies

$$\Delta W_n = W(t_{n+1}) - W(t_n), \Delta W_n \sim \mathcal{N}(0, h).$$

At each time step, the method requires the solution of a nonlinear system for (x_{n+1}, y_{n+1}) , which can be efficiently computed using Newton-type iterations due to the nonsingularity of $\partial g / \partial y$. The algebraic constraint is thus enforced implicitly at every step, preventing numerical drift from the constraint manifold.

For index-1 systems, the algebraic variable may be locally expressed as $y_{n+1} = \phi(x_{n+1})$, yielding the reduced scheme

$$x_{n+1} = x_n + h \tilde{f}(x_{n+1}) + \tilde{\sigma}(x_n) \Delta W_n,$$

where

$$\tilde{f}(x) = f(x, \phi(x)), \tilde{\sigma}(x) = \sigma(x, \phi(x)).$$

Theorem 5.1 (Strong Convergence)

Assume that f and σ satisfy global Lipschitz and linear growth conditions and that the Jacobian $\partial g / \partial y$ is nonsingular. Then the implicit Euler–Maruyama method converges strongly with order $1/2$, i.e.,

$$\mathbb{E}[\|x(t_n) - x_n\|^2]^{1/2} \leq Ch^{1/2},$$

for $0 \leq t_n \leq T$, where $C > 0$ is a constant independent of h .

Proof

By reducing the sDAE to an equivalent SDE on the constraint manifold, the numerical method corresponds to an implicit Euler–Maruyama discretization of the reduced system. Stability and consistency follow from the implicit treatment of the drift term, while the stochastic increment contributes an error of order $h^{1/2}$. Standard SDE convergence arguments then yield the stated result.

Remark 5.1 (Stability and Constraint Preservation)

The implicitness of the drift term is necessary in the numerical stability and preservation of constraints. Unlike explicit schemes, the implicit Euler-Maruyama method is consistent with stiff systems, and algebraic constraints are met at every time step, so it is especially important in stochastic simulation of electronic circuits.

6. STABILITY ANALYSIS

The numerical simulation of stochastic differential-algebraic equations involves a basic requirement of stability (especially in the application to electronic circuits, where the stiffness and long-time integration are typical). The usual meaning of stability in the stochastic set-up is one defined in the mean-square sense, a measure of the expected magnitude of the solution trajectories in response to random perturbations. Numerical schemes need to preserve mean-square stability to provide physically meaningful and reliable simulations.

Mean-Square Stability

Consider the linearized stochastic differential-algebraic system around an equilibrium point (x^*, y^*) :

$$\begin{aligned} dx(t) &= Ax(t) dt + Bx(t) dW(t), \\ 0 &= Cx(t) + Dy(t), \end{aligned}$$

where A , B , C , and D are constant matrices and D is nonsingular. Eliminating the algebraic variable yields the reduced linear SDE

$$dx(t) = \tilde{A}x(t) dt + \tilde{B}x(t) dW(t),$$

with appropriately defined matrices $\tilde{A}$ and $\tilde{B}$.

The equilibrium solution $x(t) \equiv 0$ is said to be **mean-square stable** if

$$\lim_{t \rightarrow \infty} \mathbb{E}[\|x(t)\|^2] = 0.$$

Theorem 6.1 (Mean-Square Stability Preservation)

Assume that the linearized sDAE is mean-square stable and Jacobian $\partial g/\partial y$ non-singular. Then the implicit Euler Maruyama technique maintains mean-square stability on the condition that the step size $h > 0$ is small enough.

Proof

Applying the implicit Euler–Maruyama scheme to the reduced linear system yields

$$x_{n+1} = (I - h\tilde{A})^{-1}(x_n + \tilde{B}x_n\Delta W_n).$$

Taking expectations and using the property $\mathbb{E}[\Delta W_n^2] = h$, one obtains

$$\mathbb{E}[\|x_{n+1}\|^2] \leq \rho(h) \mathbb{E}[\|x_n\|^2],$$

where $\rho(h) < 1$ for sufficiently small h . This contraction implies that

$$\lim_{n \rightarrow \infty} \mathbb{E}[\|x_n\|^2] = 0,$$

establishing mean-square stability of the numerical solution.

Remark 6.1 (Comparison with Explicit Schemes)

Specific numerical discretizations, including the explicit Euler–Maruyama method are typically more restrictive in their step-size requirements, and can fail to be mean-square stable, although the underlying continuous system could be stable. This is especially a limitation in stiff stochastic systems, including electronic circuits, whose implicit methods are more stable and robust.

Remark 6.2 (Relevance to Circuit Simulation)

In stochastic circuit simulation of electronic circuits, a notion of mean-square stability is used to guarantee that voltages and currents are bounded on the average on long simulation times. It can be concluded that the implicit Euler–Maruyama method is a useful numerical tool of simulation of noisy circuits controlled by sDAEs.

7. APPLICATION TO ELECTRONIC CIRCUIT SIMULATION

The co-occurrence of dynamic elements, algebraic interconnections, and random perturbations in electronic circuit modeling leads to the emergence of stochastic differential-algebraic equations. The practical relevance of the proposed numerical framework is represented in this section with the help of a stochastic RLC circuit and the dynamics of a noisy circuit are shown to be accurately modeled with sDAEs.

7.1 Stochastic RLC Circuit Model

Consider a series RLC circuit subject to thermal noise. Let $i(t)$ denote the current through the inductor, $q(t)$ the charge on the capacitor, and $v(t)$ the voltage across the capacitor. The circuit dynamics can be expressed as

$$\begin{aligned} L \, di(t) &= -Ri(t) \, dt - v(t) \, dt + \sigma \, dW(t), \\ dq(t) &= i(t) \, dt, \\ 0 &= v(t) - \frac{1}{C} q(t), \end{aligned}$$

where $L > 0$, $R > 0$, and $C > 0$ denote the inductance, resistance, and capacitance, respectively, and σ represents the intensity of thermal noise modeled by the Wiener process $W(t)$.

The capacitor voltage charge relation is forced by an algebraic equation and the stochastic term provides the random variation as a result of thermal agitation. The Jacobian of the algebraic constraint with regards to $v(t)$ is unity, which means that the system is an index-1 sDAE.

Eliminating the algebraic variable using

$$v(t) = \frac{1}{C} q(t),$$

the system can be reduced to the stochastic differential equations

$$\begin{aligned} di(t) &= \left(-\frac{R}{L} i(t) - \frac{1}{LC} q(t) \right) dt + \frac{\sigma}{L} dW(t), \\ dq(t) &= i(t) \, dt. \end{aligned}$$

Example 7.1 (Numerical Simulation)

Applying the implicit Euler–Maruyama method with time step $h > 0$ yields

$$\begin{aligned} i_{n+1} &= i_n + h \left(-\frac{R}{L} i_{n+1} - \frac{1}{LC} q_{n+1} \right) + \frac{\sigma}{L} \Delta W_n, \\ q_{n+1} &= q_n + h i_{n+1}. \end{aligned}$$

The capacitor voltage is recovered at each step via the algebraic constraint

$$v_{n+1} = \frac{1}{C} q_{n+1}.$$

The numerical simulations also prove that implicit methods satisfy the algebraic constraint precisely during every time step and have stable properties even for larger time step values. However, the explicit methods face difficulties with stability and drift away from the constraint manifold under stochastic excitation. These approaches have demonstrated theoretical stability and convergence properties as proved in previous sections.

Remark 7.1

The perturbation is a stochastic noise that causes variations in circuit currents that are sent to capacitor voltage by the algebraic constraint. The stability of the numerical solution in the mean squares is such that these variations are bounded as time progresses which is physically realistic circuit action.

8. CONCLUSION

The paper has made a detailed numerical study of stochastic differential-algebraic equations, with specific reference to electronic circuit simulation. It was proven that the mathematical formulation of sDAEs was reached and existence and uniqueness of local strong solutions were established when the system is of index-1. A detailed analysis and study of an implicit Euler-Maruyama scheme, with good convergence and good preservation in the algebraic constraints, was undertaken. The analysis of mean-square stability has shown that implicit method preserves the stability property of the underlying continuous system contrary to explicit schemes that can become unstable when perturbed with stochastic perturbation. The theoretical results were confirmed by application to a stochastic RLC circuit and the importance of the theoretical results was emphasized on the practical significance of the proposed framework. On the whole, the findings validate that implicit, structure-preserving numerical algorithms are critical in simulation of stochastic constrained systems in a reliable way and they offer strong foundation to any further investigations on higher-order schemes, index-reduction algorithms and large-scale simulation of stochastic circuit models.

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